ABSTRACT SIMPLICITY OF LOCALLY COMPACT KAC–MOODY GROUPS

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Abstract. In this paper, we establish that complete Kac–Moody groups over finite fields are abstractly simple. The proof makes an essential use of O. Mathieu's construction of complete Kac–Moody groups over fields. This construction has the advantage that both real and imaginary root spaces of the Lie algebra lift to root subgroups over arbitrary fields. A key point in our proof is the fact, of independent interest, that both real and imaginary root subgroups are contracted by conjugation of positive powers of suitable Weyl group elements.

1. Introduction

Let $A = (A_{ij})_{1\leq i,j\leq n}$ be a generalised Cartan matrix and let $\mathfrak{G} = \mathfrak{G}_A$ denote the associated Kac–Moody–Tits functor of simply connected type, as defined by J. Tits ([Tit87]). The value of $\mathfrak{G}$ over a field k is usually called a **minimal Kac–Moody group** of type A over k. This terminology is justified by the existence of larger groups associated with the same data, usually called **maximal** or **complete Kac–Moody groups**, and which are completions of $\mathfrak{G}(k)$ with respect to some suitable topology. One of them, introduced in [RR06], and which we will temporarily denote by $\hat{\mathfrak{G}}_A(k)$, is a totally disconnected topological group. It is moreover locally compact provided k is finite, and non-discrete (hence uncountable) as soon as A is not of finite type.

The question whether $\hat{\mathfrak{G}}_A(k)$ is (abstractly) simple for A indecomposable and k arbitrary is very natural and was explicitly addressed by J. Tits [Tit89]. Abstract simplicity results for $\hat{\mathfrak{G}}_A(k)$ over fields of characteristic 0 were first obtained in an unpublished note by R. Moody ([Moo82]). Moody's proof has been recently generalised by G. Rousseau ([Rou12, Thm.6.19]) who extended Moody's result to fields k of positive characteristic p that are not algebraic over $\mathbf{F}_p$. The abstract simplicity of $\hat{\mathfrak{G}}_A(k)$ when k is a finite field was shown in [CER08] in some important special cases, including groups of 2-spherical type over fields of order at least 4, as well as some other hyperbolic types under additional restrictions on the order of the ground field.

In this paper, we establish the abstract simplicity of complete Kac–Moody groups $\hat{\mathfrak{G}}_A(k)$ of indecomposable type over arbitrary finite fields, without any restriction. Our proof relies on an approach which is completely different from the one used in [CER08].

Theorem A. *Let $\hat{\mathfrak{G}}_A(\mathbf{F}_q)$ be a complete Kac–Moody group over a finite field $\mathbf{F}_q$, with generalised Cartan matrix A. Assume that A is indecomposable of indefinite type. Then $\hat{\mathfrak{G}}_A(\mathbf{F}_q)$ is abstractly simple.*

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As it turns out, it does not matter which completion of $\mathfrak{G}_A(\mathbf{F}_q)$ we are considering, see Theorem B and Remark 1 below.

After completion of this work, I was informed by Bertrand Rémy that, in a recent joint work [CR13] with I. Capdeboscq, they obtained independently a special case of this theorem, namely the abstract simplicity over finite fields of order at least 4 and of characteristic p in case p is greater than $M = \max_{i\neq j} |A_{ij}|$. Their approach is similar to the one used in [CER08].

Note that the topological simplicity of $\hat{\mathfrak{G}}_A(\mathbf{F}_q)$ (that is, the absence of nontrivial closed normal subgroups), which we will use in our proof of Theorem A, was previously established by B. Rémy when $q > 3$ (see [Rém04, Thm.2.A.1]); the tiniest finite fields were later covered by P-E. Caprace and B. Rémy (see [CR09, Prop.11]).

Note also that for incomplete groups, abstract simplicity fails in general since groups of affine type admit numerous congruence quotients. However, it has been shown by P-E. Caprace and B. Rémy ([CR09]) that $\mathfrak{G}_A(\mathbf{F}_q)$ is abstractly simple provided A is indecomposable, $q > n > 2$ and A is not of affine type. They also recently covered the rank 2 case for matrices A of the form $A = \left(\begin{smallmatrix} 2 & -m \\ -1 & 2 \end{smallmatrix}\right)$ with $m > 4$ (see [CR12, Theorem 2]).

As mentioned at the beginning of this introduction, different completions of $\mathfrak{G}(k)$ were considered in the literature, and therefore all deserve the name of "complete Kac–Moody groups". We now proceed to review them briefly.

Essentially three such completions have been constructed so far, from very different points of view. The first construction, due to B. Rémy and M. Ronan ([RR06]), is the one we considered above. It is the completion of the image of $\mathfrak{G}(k)$ in the automorphism group $\mathrm{Aut}(X_+)$ of its associated positive building X_+, where $\mathrm{Aut}(X_+)$ is equipped with the compact-open topology. For the rest of this paper, we will denote this group by $\mathfrak{G}^{rr}(k)$, so that $\hat{\mathfrak{G}}(k) = \mathfrak{G}^{rr}(k)$ in our previous notation. To avoid taking a quotient of $\mathfrak{G}(k)$, a variant of this group has also been considered by P-E. Caprace and B. Rémy ([CR09, 1.2]). This latter group, here denoted $\mathfrak{G}^{crr}(k)$, contains $\mathfrak{G}(k)$ as a dense subgroup and admits $\mathfrak{G}^{rr}(k)$ as a quotient.

The second construction, due to L. Carbone and H. Garland ([CG03]), associates to a regular dominant integral weight λ the completion, here denoted $\mathfrak{G}^{cg\lambda}(k)$, of $\mathfrak{G}(k)$ for the so-called weight topology.

The third construction, of which we will make an essential use, was first introduced by O. Mathieu ([Mat88b, XVIII §2], [Mat88a], [Mat89, I and II]) and further developed by G. Rousseau ([Rou12]). It is more algebraic and closer in spirit to the construction of $\mathfrak{G}$. In fact, one gets a group functor over the category of $\mathbf{Z}$-algebras, which we will subsequently denote by $\mathfrak{G}^{pma}$. As noted in [Rou12, 3.20], this functor is a generalisation of the complete Kac–Moody group over $\mathbf{C}$ constructed by S. Kumar ([Kum02, 6.1.6]). Note that in this case the closure $\overline{\mathfrak{G}(k)}$ of $\mathfrak{G}(k)$ in $\mathfrak{G}^{pma}(k)$ need not be the whole of $\mathfrak{G}^{pma}(k)$. However, $\overline{\mathfrak{G}(k)} = \mathfrak{G}^{pma}(k)$ as soon as the characteristic of k is zero or greater than the maximum M (in absolute value) of the non-diagonal entries of A (see [Rou12, Proposition 6.11]).

These three constructions are strongly related, and hopefully equivalent. In particular, they all possess refined Tits systems whose associated (positive) building is the positive building X_+ of $\mathfrak{G}(k)$ (with possibly different apartment systems). Moreover, there are natural continuous group homomorphisms $\overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cg\lambda}(k)$

and $\mathfrak{G}^{cg\lambda}(k) \to \mathfrak{G}^{crr}(k)$ extending the identity on $\mathfrak{G}(k)$ (see [Rou12, 6.3]). Their composition ϕ: $\overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cg\lambda}(k) \to \mathfrak{G}^{crr}(k)$ is an isomorphism of topological groups in many cases (see [Rou12, Théorème 6.12]) and conjecturally in all cases.

If G is either $\mathfrak{G}^{pma}(k)$ or $\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$, we let $Z'(G)$ denote the kernel of the G-action on X_+. As mentioned in Remark 1 below, Theorem A immediately implies the abstract simplicity of $G/Z'(G)$ whenever G is one of $\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$ (and k is finite). As pointed out to me by Pierre-Emmanuel Caprace, our arguments in fact also imply the abstract simplicity of $\mathfrak{G}^{pma}(k)/Z'(\mathfrak{G}^{pma}(k))$, even when $\overline{\mathfrak{G}(k)} \neq \mathfrak{G}^{pma}(k)$:

Theorem B. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type. Then $\mathfrak{G}_A^{pma}(\mathbf{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$ is abstractly simple over any finite field $\mathbf{F}_q$.*

Note that even the topological simplicity of $\mathfrak{G}_A^{pma}(\mathbf{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$ was not previously known in full generality (see [Rou12, Lemme 6.14 and Proposition 6.16] for known results).

While the construction of Rémy–Ronan is more geometric in nature, the construction of Mathieu–Rousseau is purely algebraic and hence *a priori* more suitable to establish algebraic properties of complete Kac–Moody groups. The present paper is a good illustration of this idea, and we hope it provides a good motivation for studying these "algebraic completions" further.

Remark 1. When the field k is finite, the several group homomorphisms $\overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cg\lambda}(k) \to \mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k) \leq \mathrm{Aut}(X_+)$ are all surjective (see [Rou12, 6.3]), and if G is either $\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$, the effective quotient of G by the kernel $Z'(G)$ of its action on X_+ coincides with $\mathfrak{G}^{rr}(k)$. If moreover the characteristic p of k is greater than the maximum M (in absolute value) of the non-diagonal entries of A, one has $\overline{\mathfrak{G}(k)} = \mathfrak{G}^{pma}(k)$, and thus in that case there is only one simple group $G/Z'(G)$. Hence Theorem B is a consequence of Theorem A when $p > M$. If $p \leq M$, it is possible that the effective quotient of $\mathfrak{G}^{pma}(k)$ inside $\mathrm{Aut}(X_+)$ properly contains $\mathfrak{G}^{rr}(k)$ (see Remark 2 below). When this happens, Theorem B thus asserts the abstract simplicity of a different group than the one considered in Theorem A.

Finally, we notice that, although we assumed the Kac–Moody groups to be of simply connected type to simplify the notations, the results remain valid for an arbitrary Kac–Moody root datum (see Remark 6.3).

Along the proof of Theorems A and B, we establish other results of independent interest, which we now proceed to describe.

Let k be an arbitrary field. Fix a realisation of the generalised Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ as in [Kac90, §1.1]. Let $Q = \sum_{i=1}^n \mathbf{Z}\alpha_i$ be the associated root lattice, where $\alpha_1, \dots, \alpha_n$ are the simple roots. Let also Δ (respectively, $\Delta_\pm$) be the set of roots (respectively, positive/negative roots), so that $\Delta = \Delta_+ \sqcup \Delta_-$. Write also Δ^{re} and Δ^{im} (respectively, Δ_+^{re} and Δ_+^{im}) for the set of (positive) real and imaginary roots.

Recall that a subset Ψ of Δ is **closed** if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta$. For a closed subset Ψ of Δ_+, define the sub-group scheme $\mathfrak{U}_\Psi^{ma}$ of $\mathfrak{G}^{pma}$ as in [Rou12, 3.1]. Set $\mathfrak{U}^{ma+} = \mathfrak{U}_{\Delta_+}^{ma}$. One can then define **root groups** $\mathfrak{U}_{(\alpha)}^{ma}$ in $\mathfrak{U}^{ma+}$ by setting $\mathfrak{U}_{(\alpha)}^{ma} := \mathfrak{U}_{\{\alpha\}}^{ma}$ for $\alpha \in \Delta_+^{\mathrm{re}}$ and $\mathfrak{U}_{(\alpha)}^{ma} := \mathfrak{U}_{\mathbf{N}^*\alpha}^{ma}$ for $\alpha \in \Delta_+^{\mathrm{im}}$.

We also let $\mathfrak{B}^+$, $\mathfrak{U}^+$, $\mathfrak{N}$ and $\mathfrak{T}$ denote, as in [Rou12, 1.6], the sub-functors of $\mathfrak{G} = \mathfrak{G}_A$ such that over k, $(\mathfrak{B}^+(k) = \mathfrak{U}^+(k) \rtimes \mathfrak{T}(k), \mathfrak{N}(k))$ is the canonical positive BN-pair attached to $\mathfrak{G}(k)$, and $\mathfrak{N}(k)/\mathfrak{T}(k) \cong W$, where $W = W(A)$ is the Coxeter group attached to A. We fix once for all a section $W \cong \mathfrak{N}(k)/\mathfrak{T}(k) \to \mathfrak{N}(k) : w \mapsto \overline{w}$. Note that $\mathfrak{N}$ can be viewed as a sub-functor of G^{pma} (see [Rou12, 3.12, Remarque 1]).

Finally, given a topological group H and an element $a \in H$, we define the **contraction group** $\mathrm{con}^H(a)$, or simply $\mathrm{con}(a)$, as the set of elements $g \in H$ such that $a^n g a^{-n} \overset{n\to\infty}{\to} 1$. Note then that for any $a \in \overline{\mathfrak{G}(k)} \subseteq \mathfrak{G}^{pma}(k)$, one has $\varphi(\mathrm{con}^{\mathfrak{G}^{pma}(k)}(a) \cap \overline{\mathfrak{G}(k)}) \subseteq \mathrm{con}^{\mathfrak{G}^{rr}(k)}(\varphi(a))$, where we denote by φ the composition $\overline{\mathfrak{G}(k)} \xrightarrow{\phi} \mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k)$.

Theorem C. *Let k be an arbitrary field.*

(1) *Let $w \in W$ and let $\Psi \subseteq \Delta_+$ be a closed set of positive roots such that $w\Psi \subseteq \Delta_+$. Then $\overline{w}\mathfrak{U}^{ma}_{\Psi}\overline{w}^{-1} = \mathfrak{U}^{ma}_{\omega\Psi}$.*
(2) *Let $w \in W$ and $\alpha \in \Delta_+$ be such that $w^l\alpha$ is positive and different from α for all positive integers l. Then $\mathfrak{U}^{ma}_{(\alpha)} \subseteq \mathrm{con}^{\mathfrak{G}^{pma}(k)}(\overline{w})$. In particular $\varphi(\mathfrak{U}^{ma}_{(\alpha)} \cap \overline{\mathfrak{G}(k)}) \subseteq \mathrm{con}^{\mathfrak{G}^{rr}(k)}(\overline{w})$.*
(3) *Assume that A is of indefinite type. Then there exists some $w \in W$ such that $\mathfrak{U}^{ma}_{(\alpha)} \subseteq \mathrm{con}^{\mathfrak{G}^{pma}(k)}(\overline{w}) \cup \mathrm{con}^{\mathfrak{G}^{pma}(k)}(\overline{w}^{-1})$ for all $\alpha \in \Delta_+$. Hence root groups (associated to both real and imaginary roots) are contracted.*

The proof of Theorem C can be found at the end of Section 4. The idea to prove Theorem A once Theorem C is established is the following. As the topological simplicity of $\mathfrak{G}^{rr}(\mathbf{F}_q)$ is known, it suffices to consider a dense normal subgroup K of $\mathfrak{G}^{rr}(\mathbf{F}_q)$. We then find an element $a \in \varphi^{-1}(K) \subseteq \mathfrak{G}^{pma}(\mathbf{F}_q)$ such that $\mathfrak{U}^{ma}_{(\alpha)}(\mathbf{F}_q) \subseteq \mathrm{con}^{\mathfrak{G}^{pma}(\mathbf{F}_q)}(a) \cup \mathrm{con}^{\mathfrak{G}^{pma}(\mathbf{F}_q)}(a^{-1})$ for all $\alpha \in \Delta_+$ as in Theorem C (3). We deduce that $\mathfrak{U}^{rr+}(\mathbf{F}_q)$ is contained in the subgroup generated by the closures of $\mathrm{con}^{\mathfrak{G}^{rr}(\mathbf{F}_q)}(\varphi(a)^{\pm 1})$. We then conclude by invoking the following result due to Caprace–Reid–Willis, whose proof is included in an appendix (see Appendix A).

Theorem (Caprace–Reid–Willis)**.** *Let G be a totally disconnected locally compact group and let $f \in G$. Any abstract normal subgroup of G containing f also contains the closure $\overline{\mathrm{con}(f)}$.*

The proof of Theorem B follows the exact same lines, except that in this case the topological simplicity of the group is not known in full generality, and we need one more argument to establish it.

We also point out that Theorem C has another application concerning the existence of non-closed contraction groups in complete Kac–Moody groups of non-affine type. Recall that in simple algebraic groups over local fields, contraction groups are always closed (they are in fact either trivial, or coincide with the unipotent radical of some parabolic subgroup). In particular they are closed in a complete Kac–Moody group G over a finite field as soon as the defining generalised Cartan matrix A is of affine type. It has been shown in [BRR08] that, on the other hand, if A is indecomposable non spherical, non affine and of size at least 3, then the contraction group $\mathrm{con}(a)$ of some element $a \in G$ must be non-closed. The following result shows that this also holds when A is indecomposable non spherical, non affine and of size 2.

Theorem D. *Let A denote an $n \times n$ generalised Cartan matrix of indecomposable indefinite type, let $W = W(A)$ be the associated Weyl group, and let $w = s_1 \dots s_n$ denote the Coxeter element of W. Let also G be one the complete Kac–Moody groups $\mathfrak{G}^{rr}_A(\mathbf{F}_q)$ or $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$ of simply connected type. Then the contraction group $\overline{\mathrm{con}^G(w)}$ is not closed in G, unless maybe if $G = \mathfrak{G}^{rr}_A(\mathbf{F}_q)$ and $\mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_q) \cap \overline{\mathfrak{G}(\mathbf{F}_q)}$ is contained in the kernel of φ.*

Finally, here is a last application of our results concerning isomorphism classes of Kac–Moody groups and their completions. While over infinite fields, it is known that two minimal Kac–Moody groups can be isomorphic only if their ground field are isomorphic and their underlying generalised Cartan matrix coincide up to a row-column permutation (see [Cap09, Theorem A]), this fails to be true over finite fields. Indeed, over a given finite field, two minimal Kac–Moody groups associated with two different generalised Cartan matrices of size 2 can be isomorphic, as noticed in [Cap09, Lemma 4.3]. The following result shows that, however, the corresponding Mathieu–Rousseau completions should not be expected to be isomorphic as topological groups.

Corollary E. *There exist minimal Kac–Moody groups $G_1 = \mathfrak{G}_{A_1}(\mathbf{F}_3)$, $G_2 = \mathfrak{G}_{A_2}(\mathbf{F}_3)$ over $\mathbf{F}_3$ associated to 2×2 generalised Cartan matrices A_1, A_2, such that G_1 and G_2 are isomorphic as abstract groups, but their Mathieu–Rousseau completions $\mathfrak{G}^{pma}_{A_1}(\mathbf{F}_3)$ and $\mathfrak{G}^{pma}_{A_2}(\mathbf{F}_3)$ are not isomorphic as topological groups.*

Corollary E should be viewed as an instance of a phenomenon that holds most probably in a greater level of generality, i.e. for other types of Kac–Moody groups over fields of possibly larger order.

The proof of these statements will be given in Section 5.

Remark 2. The isomorphism between the minimal Kac–Moody groups G_1 and G_2 in Corollary E is the one provided in [Cap09, Lemma 4.3], and it maps the twin BN-pair of G_1 to that of G_2. In particular, the Rémy–Ronan completions $\mathfrak{G}^{rr}_{A_1}(\mathbf{F}_3)$ of G_1 and $\mathfrak{G}^{rr}_{A_2}(\mathbf{F}_3)$ of G_2 are isomorphic as topological groups. Thus $\mathfrak{G}_{A_i}(\mathbf{F}_3)$ cannot be dense in $\mathfrak{G}^{pma}_{A_i}(\mathbf{F}_3)$ for both $i = 1, 2$, for otherwise the surjective continuous homomorphisms $\varphi_i \colon \mathfrak{G}^{pma}_{A_i}(\mathbf{F}_3) \to \mathfrak{G}^{rr}_{A_i}(\mathbf{F}_3)$ would yield isomorphisms of topological groups

$$\mathfrak{G}^{pma}_{A_1}(\mathbf{F}_3)/Z'_1 \cong \mathfrak{G}^{rr}_{A_1}(\mathbf{F}_3) \cong \mathfrak{G}^{rr}_{A_2}(\mathbf{F}_3) \cong \mathfrak{G}^{pma}_{A_2}(\mathbf{F}_3)/Z'_2$$

where $Z'_i := Z'(\mathfrak{G}^{pma}_{A_i}(\mathbf{F}_3))$, contradicting the fact that the quotients $\mathfrak{G}^{pma}_{A_1}(\mathbf{F}_3)/Z'_1$ and $\mathfrak{G}^{pma}_{A_2}(\mathbf{F}_3)/Z'_2$ are not isomorphic as topological groups (see the proof of Corollary E in Section 5). Thus Corollary E yields other examples of minimal Kac–Moody groups that are not dense in their Mathieu–Rousseau completions, besides the ones given over $\mathbf{F}_2$ in [Rou12, 6.10].

The paper is organised as follows. We first fix some notations in Section 2, and prove some preliminary results about the Coxeter group W and the set of roots Δ in Section 3. We then use these results to prove a more precise version of Theorem C in Section 4. We establish its consequences in Section 5, and we conclude the proof of Theorems A and B in Section 6.

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2. Basic facts and notations

For the rest of this paper, k denotes an arbitrary field and $A = (a_{ij})_{1\leq i,j\leq n}$ denotes a generalised Cartan matrix. We fix a realisation of A as in [Kac90, §1.1]. We then keep all notations from the introduction. In particular, Δ is the corresponding set of roots and $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. For $\alpha \in \Delta$, we denote by $\mathrm{ht}(\alpha)$ its height.

Recall also the definitions of the incomplete Kac–Moody group $\mathfrak{G}(k)$ (respectively, complete Kac–Moody groups $\mathfrak{G}^{rr}(k)$ and $\overline{\mathfrak{G}(k)} \subseteq \mathfrak{G}^{pma}(k)$) and of the sub-functors $\mathfrak{B}^+$, $\mathfrak{U}^+$, $\mathfrak{N}$ and $\mathfrak{T}$ of $\mathfrak{G}$ (respectively, and of the sub-group schemes $\mathfrak{U}^{ma}_{\Psi}$, $\mathfrak{U}^{ma}_{(\alpha)}$ and $\mathfrak{U}^{ma+}$ of $\mathfrak{G}^{pma}$). In addition, we set $\mathfrak{B}^{ma+} = \mathfrak{T} \ltimes \mathfrak{U}^{ma+}$ (see [Rou12, 3.5]) and $\mathfrak{U}^{ma}_n := \mathfrak{U}^{ma}_{\Psi(n)}$, where $\Psi(n) = \{\alpha \in \Delta^+ \mid \mathrm{ht}(\alpha) \geq n\}$. We also define the subgroups $\mathfrak{U}^{rr+}(k)$ and $\mathfrak{B}^{rr+}(k)$ of $\mathfrak{G}^{rr}(k)$ as the closures in $\mathfrak{G}^{rr}(k)$ of $\mathfrak{U}^+(k)$ and $\mathfrak{B}^+(k)$, respectively. We recall that $(\mathfrak{B}^{rr+}(k), \mathfrak{N}(k))$ is a BN-pair for $\mathfrak{G}^{rr}(k)$ (see [CR09, Proposition 1]) and that $(\mathfrak{B}^{ma+}, \mathfrak{N}(k))$ is a BN-pair for $\mathfrak{G}^{pma}$ (see [Rou12, 3.16]).

Lemma 2.1. *The subgroups $\mathfrak{U}^{ma}_n(k)$ for $n \in \mathbf{N}$ form a basis of neighbourhoods of the identity in $\mathfrak{G}^{pma}(k)$.*

Proof. See [Rou12, 6.3.6]. □

To avoid cumbersome notation, we will write $\mathrm{con}(a)$ for both contraction groups $\mathrm{con}^{\mathfrak{G}^{pma}(k)}(a)$ and $\mathrm{con}^{\mathfrak{G}^{rr}(k)}(a)$, as k is fixed and as it will be always clear in which group we are working.

As before, $W = W(A) \cong \mathfrak{N}(k)/\mathfrak{T}(k)$ is the Coxeter group associated to A, with generating set $S = \{s_1, \dots, s_n\}$ such that $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $i, j \in \{1, \dots, n\}$, and we fix a section $W \cong \mathfrak{N}(k)/\mathfrak{T}(k) \to \mathfrak{N}(k) : w \mapsto \overline{w}$.

Finally, we let again $\varphi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{rr}(k)$ denote the continuous homomorphism introduced in [Rou12, 6.3] (or more precisely, the composition of this homomorphism with the natural projection $\mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k)$), and we write $\overline{\mathfrak{U}^+(k)}$ for the closure of $\mathfrak{U}^+(k)$ in $\mathfrak{U}^{ma+}(k)$.

Lemma 2.2. *The kernel of φ is contained in $\mathfrak{T}(k) \ltimes \overline{\mathfrak{U}^+(k)}$ and the restriction of φ to $\overline{\mathfrak{U}^+(k)}$ is surjective onto $\mathfrak{U}^{rr+}(k)$ when the field k is finite.*

Proof. See [Rou12, 6.3]. □

3. Coxeter groups and root systems

In this section, we prepare the ground for the proof of Theorem A by establishing several results which concern the Coxeter group W and the set of roots Δ. Basics on these two topics are covered in [AB08, Chapters 1–3] and [Kac90, Chapters 1–5], respectively. We will also make use of CAT(0) realisations of (thin) buildings; basics about CAT(0) spaces can be found in [BH99].

Davis realisation. Recall from [Dav98] that any building Δ admits a metric realisation, here denoted $|\Delta|$, which is a complete CAT(0) cell complex. Moreover any group of type-preserving automorphisms of Δ acts in a canonical way by cellular isometries on $|\Delta|$. Finally, the cell supporting any point of $|\Delta|$ determines a unique spherical residue of Δ, called its **support**.

Throughout this section, we let $\Sigma = \Sigma(W, S)$ denote the Coxeter complex of W; hence $|\Sigma|$ in our notation is the Davis complex of W. Also, we let C_0 be the fundamental chamber of Σ. Finally, with the exception of Lemma 3.2 below where no particular assumption on W is made, we will always assume that W is infinite irreducible. Note that this is equivalent to saying that A is indecomposable of non-finite type.

Lemma 3.1. *Let X be a* CAT(0) *space and let $x \in X$. Let $g \in \mathrm{Isom}(X)$ be such that $\mathrm{d}(x, g^2x) = 2\,\mathrm{d}(x, gx) > 0$. Then g is a hyperbolic isometry and $D := \bigcup_{n\in\mathbf{Z}} [g^n x, g^{n+1}x] \subset X$ is an axis for g, where $[y, z]$ denotes the unique geodesic joining the points y and z in X.*

Proof. Since D is g-invariant, we only have to check that it is a geodesic. Set $d := \mathrm{d}(x, gx)$. We prove by induction on $n + m$, $n, m \in \mathbf{N}$, that $D_{n,m} := \bigcup_{-n\leq l\leq m+1} [g^l x, g^{l+1}x] \subset D$ is a geodesic. For $n = m = 0$, this is the hypothesis. Let now $n, m \geq 0$ and let us prove that $D_{n,m+1}$ is a geodesic (the proof for $D_{n+1,m}$ being identical). By the CAT(0) inequality applied to the triangle $A = g^{-n}x$, $B = g^m x$, $C = g^{m+1}x$, we get that

$$\mathrm{d}^2(M, C) \leq \frac{1}{2}(\mathrm{d}^2(A, C)+\mathrm{d}^2(B, C))-\frac{1}{4}\,\mathrm{d}^2(A, B) = \frac{1}{2}(\mathrm{d}^2(A, C)+d^2)-\frac{1}{4}(m+n)^2 d^2,$$

where M is the midpoint of $[A, B]$. Since by induction hypothesis $\mathrm{d}(M, C) = \frac{1}{2}(m + n)d + d$, we finally get that $\mathrm{d}^2(A, C) \geq d^2(m + n + 1)^2$, as desired.

We remark that this lemma is immediate using the notion of angle in a CAT(0) space; we prefered however to give a more elementary argument here. □

Lemma 3.2. *Let $w = s_1 \dots s_n$ be the Coxeter element of W. Let $A = A_1 + A_2$ be the unique decomposition of A as a sum of matrices A_1, A_2 such that A_1 (respectively, A_2) is an upper (respectively, lower) triangular matrix with 1's on the diagonal. Then the matrix of w in the basis $\{\alpha_1, \dots, \alpha_n\}$ of simple roots is $-A_1^{-1}A_2 = I_n - A_1^{-1}A$.*

Proof. For a certain property P of two integer variables i, j (e.g. $\mathrm{P}(i, j) \equiv j \leq i$), we introduce for short the Kronecker symbol $\delta_{\mathrm{P}(i,j)}$ taking value 1 if $\mathrm{P}(i, j)$ is satisfied and 0 otherwise.

Let $B = (b_{ij})$ denote the matrix of w in the basis $\{\alpha_1, \dots, \alpha_n\}$. Thus, b_{ij} is the coefficient of α_i in the expression of $s_1 \dots s_n\alpha_j$ as a linear combination of the simple roots, which we will write for short as $[s_1 \dots s_n\alpha_j]_i$. Thus $b_{ij} = [s_1 \dots s_n\alpha_j]_i = [s_i \dots s_n\alpha_j]_i$. Note that

$$s_{i+1} \dots s_n\alpha_j = \sum_{k=i+1}^{n} [s_{i+1} \dots s_n\alpha_j]_k\alpha_k + \delta_{j\leq i}\alpha_j = \sum_{k=i+1}^{n} b_{kj}\alpha_k + \delta_{j\leq i}\alpha_j.$$

Whence

$$\begin{aligned}
b_{ij} &= [s_i(\sum_{k=i+1}^{n} b_{kj}\alpha_k + \delta_{j\leq i}\alpha_j)]_i = -\sum_{k=i+1}^{n} a_{ik}b_{kj} - \delta_{j\leq i}a_{ij} + \delta_{i=j}\\
&= (-\sum_{k=1}^{n}(A_1)_{ik}b_{kj} + b_{ij}) + (\delta_{j>i}a_{ij} - a_{ij}) + \delta_{i=j}\\
&= -\sum_{k=1}^{n}(A_1)_{ik}b_{kj} + b_{ij} - a_{ij} + \sum_{k=1}^{n}(A_1)_{ik}(I_n)_{kj}.
\end{aligned}$$

Thus $A = -A_1B + A_1$, so that $B = -A_1^{-1}A_2$, as desired. □

Lemma 3.3. *Let $w = s_1 \dots s_n$ be the Coxeter element of W. Then w acts on $|\Sigma|$ as a hyperbolic isometry. Moreover, there exists some $v \in W$ such that $w_1 := vwv^{-1}$ possesses an axis D going through some point $x_0 \in |\Sigma|$ whose support is C_0. In particular, x_0 does not lie on any wall of $|\Sigma|$.*

Proof. Note first that w is indeed hyperbolic, for otherwise it would be elliptic by [Bri99] and hence would be contained in a spherical parabolic subgroup of W, contradicting the fact that its parabolic closure is the whole of W (see [Par07, Theorem 3.4]).

Note also that w does not stabilise any wall of $|\Sigma|$. Indeed, suppose to the contrary that there exists some positive real root $\alpha \in \Delta_+$ such that $w\alpha = \pm\alpha$. Recall the decomposition $A = A_1 + A_2$ from Lemma 3.2. Viewing w as an automorphism of the root lattice, it follows from this lemma that $A_2\alpha = \mp A_1\alpha$. If $w\alpha = \alpha$, this implies that $A\alpha = A_1\alpha + A_2\alpha = 0$, hence that α is an imaginary root by [Kac90, Lemma 5.3], a contradiction. Assume now that $w\alpha = -\alpha \in \Delta_-$. Then there is some $t \in \{1, \dots, n\}$ such that $\alpha = s_n \dots s_{t+1}\alpha_t$. Hence $w\alpha = s_1 \dots s_{t-1}(-\alpha_t)$ and thus $s_n \dots s_{t+1}\alpha_t = s_1 \dots s_{t-1}\alpha_t$. Writing these expressions in the basis $\{\alpha_1, \dots, \alpha_n\}$ yields $n = t = 1$ or $a_{it} = 0$ for all $i \neq t$, contradicting the fact that W is infinite irreducible.

Therefore, for any wall m of $|\Sigma|$ and any w-axis D, either $m \cap D$ is empty or consists of a single point (see [NV02, Lemma 3.4]). Thus any w-axis contains a point which does not belong to any wall. Since the W-action is transitive on the chambers, the conclusion follows. □

Lemma 3.4. *Let w_1 be as in Lemma 3.3. Let $t_1t_2 \dots t_k$ be a reduced expression for w_1, where $t_j \in S$ for all $j \in \{1, \dots, k\}$. Then for all $l \in \mathbf{N}$ and $j \in \{1, \dots, k\}$, one has $\ell(t_jt_{j+1} \dots t_kw_1^l) = \ell(t_{j+1} \dots t_kw_1^l) + 1$ and $\ell(t_jt_{j-1} \dots t_1w_1^{-l}) = \ell(t_{j-1} \dots t_1w_1^{-l}) + 1$.*

Proof. Note that since $\ell(sv) \leq \ell(v) + 1$ for $s \in S$ and $v \in W$, it is sufficient to show that $\ell(w_1^l) = l\ell(w_1) = lk$ for all $l \in \mathbf{N}^*$. Let x_0 be as in Lemma 3.3. Then $\ell(w_1^l)$ coïncides with the number of walls separating x_0 from w_1x_0 in $|\Sigma|$ (see [AB08, Lemma 3.69]). In particular, k walls separate x_0 from w_1x_0, and the claim then follows from Lemma 3.3. □

For $\omega \in W$ and $\alpha \in \Delta_+$, define the function $f_\alpha^\omega\colon \mathbf{Z} \to \{\pm 1\} : k \mapsto \mathrm{sign}(\omega^k\alpha)$, where $\mathrm{sign}(\Delta_\pm) = \pm 1$.

Lemma 3.5. *Let $w = s_1 \dots s_n$ be the Coxeter element of W, and let w_1 be as in Lemma 3.3. Then the following hold.*

(1) *Let* $\omega \in W$ *be such that* $\ell(\omega^l) = |l|\ell(\omega)$ *for all* $l \in \mathbf{Z}$. *Then* f^ω_α *is monotonic for all* $\alpha \in \Delta_+$.
(2) f^w_α *and* $f^{w_1}_\alpha$ *are monotonic for all* $\alpha \in \Delta_+$.

Proof. Let $\omega \in W$ be such that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbf{Z}$ and let $\omega = t_1t_2 \dots t_k$ be a reduced expression for ω, where $t_j \in S$ for all $j \in \{1, \dots, k\}$. Let $\alpha \in \Delta_+$ and assume that f^ω_α is not constant. Then α is a real root because $W.\Delta^{\mathrm{im}}_+ = \Delta^{\mathrm{im}}_+$. Let $k_\alpha \in \mathbf{Z}^*$ be minimal (in absolute value) so that $f^\omega_\alpha(k_\alpha) = -1$. We deal with the case when $k_\alpha > 0$; the same proof applies for $k_\alpha < 0$ by replacing ω with its inverse. We have to show that $\omega^l\alpha \in \Delta_-$ if and only if $l \geq k_\alpha$.

Let $\beta := \omega^{k_\alpha - 1}\alpha$. Thus $\beta \in \Delta^{\mathrm{re}}_+$ and $\omega\beta \in \Delta^{\mathrm{re}}_-$. It follows that there is some $i \in \{1, \dots, k\}$ such that $\beta = t_kt_{k-1} \dots t_{i+1}\alpha_{t_i}$. In other words, β is one of the n positive roots whose wall $\partial\beta$ in the Coxeter complex Σ of W separates the fundamental chamber C_0 from $\omega^{-1}C_0$. We want to show that $\omega^l\beta \in \Delta_-$ if and only if $l \geq 1$.

Assume first for a contradiction that there is some $l \geq 1$ such that $\omega^{l+1}\beta \in \Delta_+$, that is, $\omega^{l+1}\beta$ contains C_0. Since $\omega^{l+1}\beta$ contains $\omega^{l+1}C_0$ but not ω^lC_0, its wall $\omega^{l+1}\partial\beta$ separates ω^lC_0 from $\omega^{l+1}C_0$ and C_0. In particular, any gallery from C_0 to $\omega^{l+1}C_0$ going through ω^lC_0 cannot be minimal. This contradicts the assumption that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbf{Z}$ since this implies that the product of $l+1$ copies of $t_1 \dots t_k$ is a reduced expression for ω^{l+1}.

Assume next for a contradiction that there is some $l \geq 1$ such that $\omega^{-l}\beta \in \Delta_-$. Then as before, $\omega^{-l}\partial\beta$ separates $\omega^{-l}C_0$ from $\omega^{-l-1}C_0$ and C_0. Again, this implies that any gallery from C_0 to $\omega^{-l-1}C_0$ going through $w^{-l}C_0$ cannot be minimal, a contradiction. This proves the first statement.

The second statement is then a consequence of the first and of [Spe09] in case $\omega = w$ (respectively, and of Lemma 3.4 in case $\omega = w_1$). ☐

Lemma 3.6. *Let* $w = s_1 \dots s_n$ *be the Coxeter element of* W. *Let* $\alpha \in \Delta_+$. *Assume that* A *is of indefinite type. Then* $w^l\alpha \neq \alpha$ *for all nonzero integer* l.

Proof. Assume for a contradiction that $w^k\alpha = \alpha$ for some $k \in \mathbf{N}^*$. It then follows from Lemma 3.5 that $w^i\alpha \in \Delta_+$ for all $i \in \{0, \dots, k-1\}$. Viewing w as an automorphism of the root lattice, we get that

$$(w - \mathrm{Id})(w^{k-1} + \dots + w + \mathrm{Id})\alpha = 0.$$

Moreover, $\beta := (w^{k-1} + \dots + w + \mathrm{Id})\alpha$ is a sum of positive roots, and hence can be viewed as a nonzero vector of $\mathbf{R}^n$ with nonnegative entries. Recall from Lemma 3.2 that w is represented by the matrix $-A_1^{-1}A_2$. Thus, multiplying the above equality by $-A_1$, we get that $A\beta = 0$. Since A is indecomposable of indefinite type, this gives the desired contradiction by [Kac90, Theorem 4.3]. ☐

Lemma 3.7. *Let* $\omega \in W$ *and* $\alpha \in \Delta_+$ *be such that* $\omega^l\alpha \neq \alpha$ *for all positive integer* l. *Then* $|\operatorname{ht}(\omega^l\alpha)|$ *goes to infinity as* l *goes to infinity.*

Proof. If $|\operatorname{ht}(\omega^l\alpha)|$ were bounded as l goes to infinity, the set of roots $\{\omega^l\alpha \mid l \in \mathbf{N}\}$ would be finite, and so there would exist an $l \in \mathbf{N}^*$ such that $\omega^l\alpha = \alpha$, a contradiction. ☐

Lemma 3.8. *Let* w_1 *be as in Lemma 3.3. Let* $\alpha \in \Delta_+$ *and let* $\epsilon \in \pm$ *be such that* $w_1^{\epsilon k}\alpha \in \Delta_+$ *for all* $k \in \mathbf{N}$. *Assume that* A *is of indefinite type. Then* $\operatorname{ht}(w_1^{\epsilon k}\alpha)$ *goes to infinity as* k *goes to infinity.*

Proof. Writing $w_1 = v^{-1}wv$ for some $v \in W$, where $w = s_1 \dots s_n$ is the Coxeter element of W, we notice that $w_1^l \alpha = \alpha$ for some integer l if and only if $w^l \beta = \beta$, where $\beta = v\alpha$. Thus the claim follows from Lemmas 3.6 and 3.7. □

4. Contraction groups

In this section, we make use of the results proven so far to establish, under suitable hypotheses, that the subgroups $\mathfrak{U}^{ma+}(k)$ of $\mathfrak{G}^{pma}(k)$ and $\mathfrak{U}^{rr+}(k)$ of $\mathfrak{G}^{rr}(k)$ are contracted. Moreover, we give some control on the contraction groups involved using building theory. Basics on this theory can be found in [AB08, Chapters 4–6].

Throughout this section, we let X_+ denote the positive building associated to $\mathfrak{G}(k)$ and we write Σ_0 and C_0 for the fundamental apartment and chamber of X_+, respectively. As before, $|X_+|$ and $|\Sigma_0|$ denote the corresponding Davis realisations. Finally, we again assume that W is infinite irreducible and we fix an element w_1 of W as in Lemma 3.3.

Lemma 4.1. *Let H be a topological group acting on a set E with open stabilisers. Then any dense subgroup of H is orbit-equivalent to H.*

Proof. Let N be a dense subgroup of H. Let x, y be two points of E in the same H-orbit, say $y = hx$ for some $h \in H$. As the stabiliser H_x of x in H is open, the open neighbourhood hH_x of h in H must intersect N, whence the result. □

Lemma 4.2. *Let G be either $\mathfrak{G}^{rr}(k)$ or $\mathfrak{G}^{pma}(k)$, and set $B := \mathfrak{B}^{rr+}(k)$ or $B := \mathfrak{B}^{ma+}(k)$ accordingly. Let K be a dense normal subgroup of G. Then there exist an element $a \in K$ and elements $b_l \in B$ for $l \in \mathbf{Z}$ such that $a^l = b_l \overline{w_1}^l$ for all $l \in \mathbf{Z}$.*

Proof. Let $x_0 \in |\Sigma_0|$ be as in Lemma 3.3. By Lemma 4.1 applied to the action of G on the set of couples of points in $|X_+|$, one can find some $a_1 \in K$ such that $a_1 \overline{w_1}^{-1} x_0 = x_0$ and $a_1 x_0 = \overline{w_1} x_0$. By Lemma 3.1 together with Lemma 3.3, we know that a_1 is hyperbolic and that $D := \bigcup_{l \in \mathbf{Z}} [a_1^l x_0, a_1^{l+1} x_0]$ is an axis of a_1. In particular, D is contained in the Davis realisation of an apartment $b\Sigma_0$ for some $b \in B$ (see [CH09, Theorem 5]). Thus $a := b^{-1} a_1 b$ is a hyperbolic element of K possessing $b^{-1} D \subseteq |\Sigma_0|$ as a translation axis.

Note that since $a_1 x_0 = \overline{w_1} x_0$ we have $aC_0 = b^{-1} \overline{w_1} C_0$ and so a belongs to the double coset $B \overline{w_1} B$. It follows from [AB08, 6.2.6] together with Lemma 3.4 that $a^l \in B \overline{w_1}^l B$ for all $l \in \mathbf{Z}$. Since $a^l C_0 \in \Sigma_0$ and hence $a^l C_0 = \overline{w_1}^l C_0$ for all $l \in \mathbf{Z}$, one can then find elements $b_l \in B$, $l \in \mathbf{Z}$, such that $a^l = \overline{w_1}^l b_{-l}^{-1}$ for all $l \in \mathbf{Z}$. Taking inverses, this yields $a^l = b_l \overline{w_1}^l$ for all $l \in \mathbf{Z}$, as desired. □

Lemma 4.3. *Let $\Psi_1 \subseteq \Psi_2 \subseteq \dots \subseteq \Delta_+$ be an increasing sequence of closed subsets of Δ_+ and set $\Psi = \bigcup_{i=1}^{\infty} \Psi_i$. Then the corresponding increasing union of subgroups $\bigcup_{i=1}^{\infty} \mathfrak{U}^{ma}_{\Psi_i}(k)$ is dense in $\mathfrak{U}^{ma}_{\Psi}(k)$.*

Proof. This follows from [Rou12, Proposition 3.2]. □

Proposition 4.4. *Let $\Psi \subseteq \Delta_+$ be closed. Let $\omega \in W$ be such that $\omega\Psi \subseteq \Delta_+$. Then $\overline{\omega} \mathfrak{U}^{ma}_{\Psi} \overline{\omega}^{-1} = \mathfrak{U}^{ma}_{\omega\Psi}$.*

Proof. For a positive root $\alpha \in \Delta_+$, recall the notations $\mathfrak{U}^{ma}_{(\alpha)} := \mathfrak{U}^{ma}_{\{\alpha\}}$ if α is real and $\mathfrak{U}^{ma}_{(\alpha)} := \mathfrak{U}^{ma}_{\mathbf{N}^*\alpha}$ if α is imaginary.

Note first that if $\alpha \in \Delta_+^{\mathrm{re}}$ is such that $w\alpha \in \Delta_+$, then $\overline{\omega} \mathfrak{U}^{ma}_{(\alpha)} \overline{\omega}^{-1} = \mathfrak{U}^{ma}_{(\omega\alpha)}$ by [Rou12, 3.11].

Let now $\beta \in \Delta_+^{\mathrm{im}}$. We claim that $\overline{\omega}\mathfrak{U}^{ma}_{(\beta)}\overline{\omega}^{\,-1} \subseteq \mathfrak{U}^{ma}_{(\omega\beta)}$ for all $w \in W$. Indeed, since $W\beta \subseteq \Delta_+^{\mathrm{im}}$, it is sufficient to show that $\overline{s_i}\mathfrak{U}^{ma}_{(\beta)}\overline{s_i}^{\,-1} \subseteq \mathfrak{U}^{ma}_{(s_i\beta)}$ for all $i \in \{1, \dots, n\}$: the claim will then follow by induction on $\ell(w)$. By [Rou12, Proposition 3.2] (and in the notations of *loc. cit.*) this amounts to show that $\overline{s_i}([\exp]x)\overline{s_i}^{\,-1} = [\exp](s_i^* x)$ for all homogenous $x \in \oplus_{n\geq 1}\mathfrak{g}_{n\beta}$, where $\mathfrak{g}$ denotes the Kac–Moody algebra of $\mathfrak{G}$. This last statement follows by definition of the semi-direct products defining the minimal parabolic subgroups of $\mathfrak{G}^{pma}$ (see [Rou12, 3.5]).

Let now Ψ and ω be as in the statement of the lemma. By the above discussion, we know that

$$\overline{\omega}\langle \mathfrak{U}^{ma}_{(\alpha)} \mid \alpha \in \Psi\rangle\overline{\omega}^{\,-1} \subseteq \langle \mathfrak{U}^{ma}_{(\omega\alpha)} \mid \alpha \in \Psi\rangle.$$

Passing to the closures, it follows from Lemma 4.3 that $\overline{\omega}\mathfrak{U}^{ma}_{\Psi}\overline{\omega}^{\,-1} \subseteq \mathfrak{U}^{ma}_{\omega\Psi}$, as desired.

We remark that this proposition is implicitely contained in [Rou12] (see [Rou12, 3.12, Remarque 2 and 6.3.2]). □

Lemma 4.5. *Let $\Psi \subseteq \Delta_+$ be the set of positive roots α such that $w_1^l\alpha \in \Delta_+$ for all $l \in \mathbf{N}$. Then both Ψ and $\Delta_+ \setminus \Psi$ are closed. In particular, one has a unique decomposition $\mathfrak{U}^{ma+} = \mathfrak{U}^{ma}_{\Psi}.\mathfrak{U}^{ma}_{\Delta_+\setminus\Psi}$.*

Proof. Clearly, Ψ is closed. Let now $\alpha, \beta \in \Delta_+ \setminus \Psi$ be such that $\alpha+\beta \in \Delta$. Thus there exist some positive integers l_1, l_2 such that $w_1^{l_1}\alpha \in \Delta_-$ and $w_1^{l_2}\beta \in \Delta_-$. Then $w_1^l(\alpha+\beta) \in \Delta_-$ for all $l \geq \max\{l_1, l_2\}$ by Lemma 3.5 and hence $\alpha+\beta \in \Delta_+ \setminus \Psi$. Thus $\Delta_+ \setminus \Psi$ is closed, as desired. The second statement follows from [Rou12, Lemme 3.3]. □

Remark 4.6. Let $\Psi \subseteq \Delta_+$ be as in Lemma 4.5. Put an arbitrary order on Δ_+. This yields enumerations $\Psi = \{\beta_1, \beta_2, \dots\}$ and $\Delta_+ \setminus \Psi = \{\alpha_1, \alpha_2, \dots\}$. For each $i \in \mathbf{N}^*$, we let Ψ_i (respectively, Φ_i) denote the closure in Δ_+ of $\{\beta_1, \dots, \beta_i\}$ (respectively, of $\{\alpha_1, \dots, \alpha_i\}$). It follows from Lemma 4.5 that $\Psi = \bigcup_{i=1}^{\infty}\Psi_i$ and that $\Delta_+ \setminus \Psi = \bigcup_{i=1}^{\infty}\Phi_i$.

Lemma 4.7. *Fix $i \in \mathbf{N}^*$, and let $\Psi_i, \Phi_i \subseteq \Delta_+$ be as in Remark 4.6. Assume that A is of indefinite type. Then there exists a sequence of positive integers $(n_k)_{k\in\mathbf{N}}$ going to infinity as k goes to infinity, such that $\overline{w_1}^{\,k}\mathfrak{U}^{ma}_{\Psi_i}\overline{w_1}^{\,-k} \subseteq \mathfrak{U}^{ma}_{n_k}$ and $\overline{w_1}^{\,-k}\mathfrak{U}^{ma}_{\Phi_i}\overline{w_1}^{\,k} \subseteq \mathfrak{U}^{ma}_{n_k}$ for all $k \in \mathbf{N}$.*

Proof. Let $\alpha_j, \beta_j \in \Delta_+$ be as in Remark 4.6. By Lemma 3.8 together with Lemma 3.5, one can find for each $j \in \{1, \dots, i\}$ sequences of positive integers $(m_k^j)_{k\in\mathbf{N}}$ and $(n_k^j)_{k\in\mathbf{N}}$ going to infinity as k goes to infinity, such that $\mathrm{ht}(w_1^{-k}\alpha_j) \geq m_k^j$ and $\mathrm{ht}(w_1^k\beta_j) \geq n_k^j$ for all $k \in \mathbf{N}$. For each $k \in \mathbf{N}$, set $n_k = \min\{m_j^k, n_j^k \mid 1 \leq j \leq i\}$. Then the sequence $(n_k)_{k\in\mathbf{N}}$ goes to infinity as k goes to infinity. Moreover, $\mathrm{ht}(\alpha) \geq n_k$ for all $\alpha \in w_1^{-k}\Phi_i$ and $\mathrm{ht}(\beta) \geq n_k$ for all $\beta \in w_1^k\Psi_i$. The conclusion then follows from Proposition 4.4. □

Theorem 4.8. *Let $a \in \mathfrak{G}^{pma}(k)$ be such that $a^l = b_l\overline{w_1}^{\,l}$ for all $l \in \mathbf{Z}$, for some $b_l \in \mathfrak{B}^{ma+}(k)$. Let Ψ, Ψ_i, Φ_i be as in Remark 4.6 and assume that A is of indefinite type. Then the following hold.*

(1) $\mathfrak{U}^{ma}_{\Psi_i}(k) \subseteq \mathrm{con}(a)$ *and* $\mathfrak{U}^{ma}_{\Phi_i}(k) \subseteq \mathrm{con}(a^{-1})$ *for all* $i \in \mathbf{N}^*$.
(2) $\mathfrak{U}^{ma}_{\Psi}(k) \subseteq \overline{\mathrm{con}(a)}$ *and* $\mathfrak{U}^{ma}_{\Delta_+\setminus\Psi}(k) \subseteq \overline{\mathrm{con}(a^{-1})}$.
(3) $\mathfrak{U}^{ma+}(k) \subseteq \langle\overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})}\rangle$.

Proof. Note that $\mathfrak{U}_n^{ma}(k)$ is normal in $\mathfrak{U}^{ma+}(k)$, and thus also in $\mathfrak{B}^{ma+}(k)$, for all $n \in \mathbf{N}$ (see [Rou12, Lemme 3.3 c)]). The first statement then follows from Lemma 4.7. The second statement is a consequence of the first together with Lemma 4.3. The third statement follows from the second together with Lemma 4.5. $\square$

Recall the definition and properties of the map φ from Section 2.

Lemma 4.9. *Let K be a dense normal subgroup of $\mathfrak{G}^{rr}(k)$. Assume that A is of indefinite type. Assume moreover that the continuous homomorphism $\varphi\colon \overline{\mathfrak{U}^+(k)} \to \mathfrak{U}^{rr+}(k)$ is surjective (e.g. k finite). Then there exists some $a \in K$ such that the following hold.*

1. *The subgroups $U_1 := \varphi(\mathfrak{U}_\Psi^{ma}(k) \cap \overline{\mathfrak{U}^+(k)})$ and $U_2 := \varphi(\mathfrak{U}_{\Delta_+\setminus\Psi}^{ma}(k) \cap \overline{\mathfrak{U}^+(k)})$ of $\mathfrak{U}^{rr+}(k)$ are respectively contained in $\overline{\mathrm{con}(a)}$ and $\overline{\mathrm{con}(a^{-1})}$.*
2. *$\mathfrak{U}^{rr+}(k) \subseteq \langle \overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})} \rangle$.*

Proof. Let $a \in K$ and $b_l \in \mathfrak{B}^{rr+}(k)$ for $l \in \mathbf{Z}$ be as in Lemma 4.2, so that $a^l = b_l\overline{w_1}^l$ for all $l \in \mathbf{Z}$. For each $l \in \mathbf{Z}$, let $\tilde{b_l} \in \mathfrak{T}(k) \ltimes \overline{\mathfrak{U}^+(k)} \subseteq \mathfrak{B}^{ma+}(k)$ be such that $\varphi(\tilde{b_l}) = b_l$. Set $\tilde{a} = \tilde{b_1}\overline{w_1} \in \overline{\mathfrak{G}(k)} \subseteq \mathfrak{G}^{pma}(k)$. Then $\varphi(\tilde{b_l}\overline{w_1}^l) = a^l = \varphi(\tilde{a}^l)$ for all $l \in \mathbf{Z}$. As the kernel of $\varphi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{rr}(k)$ lies in $\mathfrak{T}(k) \ltimes \overline{\mathfrak{U}^+(k)}$ by Lemma 2.2, we may assume up to modifying the elements $\tilde{b_l}$ that $\tilde{a}^l = \tilde{b_l}\overline{w_1}^l$ for all $l \in \mathbf{Z}$.

Since φ is continuous, both statements are then a consequence of Theorem 4.8 and of the surjectivity of $\varphi\colon \overline{\mathfrak{U}^+(k)} \to \mathfrak{U}^{rr+}(k)$. $\square$

Proof of Theorems C and B. The first statement is Proposition 4.4 and the third is contained in Theorem 4.8. The second statement is a consequence of the first together with Lemmas 3.7 and 2.1. $\square$

5. Consequences of Theorem C

Before we give the proof of Theorem A, in the next section, we examine the consequences, stated in the introduction, of Theorem C. More precisely, we will make use of the following lemma. Recall from [Wil12, Section 3] (see also the Appendix) the definition of the **nub** of an automorphism α of a totally disconnected locally compact group G. It possesses many equivalent definitions (see [Wil12, Theorem 4.12]), and given an element $a \in G$ (viewed as a conjugation automorphism), it can be characterised as $\mathrm{nub}(a) = \overline{\mathrm{con}(a)} \cap \overline{\mathrm{con}(a^{-1})}$ (see [Wil12, Remark 3.3 (b) and (d)]).

Lemma 5.1. *Let $G = \mathfrak{G}_A^{pma}(\mathbf{F}_q)$ be a complete Kac–Moody group of simply connected type over a finite field $\mathbf{F}_q$, with indecomposable generalised Cartan matrix A of indefinite type. Let $U^{im+} = \mathfrak{U}_{\Delta_+^{\mathrm{im}}}^{ma}(\mathbf{F}_q)$ denote its positive imaginary subgroup, let $w \in W = W(A)$ denote the Coxeter element of W, and set $a := \overline{w} \in \mathfrak{N}(\mathbf{F}_q)$. Then*

$$U^{im+} \subseteq \mathrm{nub}(a) = \overline{\mathrm{con}(a)} \cap \overline{\mathrm{con}(a^{-1})}.$$

Proof. Notice that Lemma 4.7 remains valid if one replaces Ψ by its subset Δ_+^{im} and w_1 by $w^{\pm 1}$. The lemma thus follows from Theorem 4.8. $\square$

To establish Theorem D, we need one more technical lemma regarding contraction groups, whose proof is an adaptation of the proof of Proposition 2.1 in [Wan84].

Lemma 5.2. *Let G be a locally compact group, let a be an element of G, and let Q be a compact subset of G such that $Q \subseteq \mathrm{con}(a)$. Then Q is uniformly contracted by a, that is, for every open neighbourhood U of the identity one has $a^nQa^{-n} \subset U$ for all large enough n.*

Proof. Fix an open neighbourhood U of the identity, and let V be a compact neighbourhood of the identity such that $V^2 \subset U$. By hypothesis, for all $x \in Q$ there exists an N_x such that $a^nxa^{-n} \in V$ for all $n \geq N_x$. In other words,

$$Q \subset \bigcup_{N\geq 0} \bigcap_{n\geq N} a^{-n}Va^n.$$

Note that the sets $C_N = \bigcap_{n\geq N} a^{-n}Va^n$ form an ascending chain of compact sets. It follows from Baire theorem that $Q \cap C_N$ has nonempty interior in Q for a large enough N.

By compacity of Q, one then finds a finite subset F of Q such that

$$Q \subset F.C_N.$$

Since F is finite and contained in $\mathrm{con}(a)$, we know that $a^nFa^{-n} \subset V$ for all large enough n. Moreover, by construction, $a^nC_Na^{-n} \subset V$ for $n \geq N$, and hence

$$a^nQa^{-n} = (a^nFa^{-n}).(a^nC_Na^{-n}) \subset V^2 \subset U$$

for all large enough n, as desired. □

Proof of Theorem D. Let A denote an $n \times n$ generalised Cartan matrix of indecomposable indefinite type, let $W = W(A)$ be the associated Weyl group, and let $w = s_1 \dots s_n$ denote the Coxeter element of W. Set $a := \overline{w} \in \mathfrak{N}(\mathbf{F}_q)$. It then follows from Lemma 5.1 that

$$U_{im}^{ma+} := \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_q) \subseteq \overline{\mathrm{con}(a)} \quad \text{in} \quad \mathfrak{G}_A^{pma}(\mathbf{F}_q)$$

and that

$$U_{im}^{rr+} := \varphi(\mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_q) \cap \overline{\mathfrak{U}^+(\mathbf{F}_q)}) \subseteq \overline{\mathrm{con}(a)} \quad \text{in} \quad \mathfrak{G}_A^{rr}(\mathbf{F}_q).$$

Since U_{im}^{ma+} is closed in $\mathfrak{U}^{ma+}(\mathbf{F}_q)$ which is compact (see [Rou12, 6.3]), both the groups U_{im}^{ma+} and U_{im}^{rr+} are compact. Moreover, they are normalised by a by Proposition 4.4. Hence they cannot be contracted by a because of Lemma 5.2, since by assumption U_{im}^{rr+} is nontrivial. In particular, $\mathrm{con}(a) \neq \overline{\mathrm{con}(a)}$ and hence $\mathrm{con}(a)$ cannot be closed.

Note that one could also directly use the fact that $\mathrm{con}(a)$ is closed if and only if $\mathrm{nub}(a) = \{1\}$ (see [Wil12, Remark 3.3 (b)]) together with Lemma 5.1. We prefered however to present a more elementary proof as well, as Lemma 5.2 will be used anyway in the proof of Corollary E below. □

Proof of Corollary E. It follows from [Cap09, Lemma 4.3] that the minimal Kac–Moody group $G_1 = \mathfrak{G}_{A_1}(\mathbf{F}_3)$ over $\mathbf{F}_3$ of simply connected type with generalised Cartan matrix $A_1 = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ (hence of affine type) is isomorphic to any minimal Kac–Moody group $G_2 = \mathfrak{G}_{A_2}(\mathbf{F}_3)$ over $\mathbf{F}_3$ of simply connected type with generalised Cartan matrix $A_2 = \left(\begin{smallmatrix} 2 & -2m \\ -2n & 2 \end{smallmatrix}\right)$ for $m, n > 1$ (hence of indefinite type). We fix such a group G_2 and we assume moreover that one of m, n (say m) is not a multiple of 3.

For $i = 1, 2$ set $\widehat{G}_i := \mathfrak{G}^{pma}_{A_i}(\mathbf{F}_3)$ and let Z'_i denote the kernel of the action of $\widehat{G}_i$ on its associated positive building. Assume for a contradiction that there is an

isomorphism $\psi\colon\ \widehat{G}_1 \to \widehat{G}_2$ of topological groups. As noticed in [Rou12, Remarque 6.20 (4)], the quotient $\widehat{G}_1/Z'_1$ is a simple algebraic group over the local field $\mathbf{F}_3(\!(t)\!)$. In particular, all the contraction groups of $\widehat{G}_1/Z'_1$ are closed. Moreover, $\psi(Z'_1)$ is the unique maximal proper normal subgroup of $\widehat{G}_2$, and it is compact. It follows that $\psi(Z'_1) = Z'_2$, for otherwise by Tits' lemma (see [AB08, Lemma 6.61]), the group $\widehat{G}_2$ would be compact, a contradiction. Hence ψ induces an isomorphism of topological groups between $\widehat{G}_1/Z'_1$ and $\widehat{G}_2/Z'_2$, so that in particular all contraction groups of $\widehat{G}_2/Z'_2$ are closed. Let $\pi\colon\ \widehat{G}_2 \to \widehat{G}_2/Z'_2$ denote the canonical projection, and let a be any element of $\widehat{G}_2$. Then

$$\pi(\overline{\mathrm{con}(a)}) \subseteq \overline{\pi(\mathrm{con}(a))} \subseteq \overline{\mathrm{con}(\pi(a))} = \mathrm{con}(\pi(a)).$$

It follows from Lemma 5.1 that the subgroup $U^+_{im} := \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_3)$ of $U^{ma+} := \mathfrak{U}^{ma}_{\Delta_+}(\mathbf{F}_3)$ in $\widehat{G}_2$ is such that

$$\pi(U^+_{im}) \subseteq \pi(\overline{\mathrm{con}(a)}) \subseteq \mathrm{con}(\pi(a))$$

for a suitably chosen $a \in \widehat{G}_2$ normalising U^+_{im}. Thus Lemma 5.2 implies that $\pi(U^+_{im}) = \{1\}$, that is, $U^+_{im} \subseteq Z'_2$. On the other hand, if Z_2 is the center of $\widehat{G}_2$, then $Z'_2 = Z_2.(Z'_2 \cap U^{ma+})$ and $Z'_2 \cap U^{ma+}$ is normal in $\widehat{G}_2$ by [Rou12, Proposition 6.4]. Since $U^+_{im} = \bigcap_{w\in W} wU^{ma+}w^{-1}$ (where the inclusion $\subseteq$ is clear and the inclusion $\supseteq$ follows from [Rou12, Proposition 3.2] together with Proposition 4.4), it follows that $U^+_{im} = Z'_2 \cap U^{ma+}$ is normal in $\widehat{G}_2$.

Let now e_1, e_2 and f_1, f_2 be the Chevalley generators of $\mathfrak{g}(A_2)$, corresponding to simple roots α_1, α_2 and their opposite. Let also $\mathcal{U}$ denote the $\mathbf{Z}$-form of the enveloping algebra of $\mathfrak{g}(A_2)$ introduced by J. Tits (see e.g. [Rou12, Section 2]), and set $\mathfrak{g}(A_2)_{\mathbf{Z}} := \mathfrak{g}(A_2) \cap \mathcal{U}$. Then $x := [e_1, e_2] \in \mathfrak{g}(A_2)_{\mathbf{Z}}$ is nonzero and $\delta := \alpha_1 + \alpha_2$ is an imaginary root (see [Kac90, Lemma 5.3]). Moreover, $\mathrm{ad}(f_1)x$ is nonzero in $\mathfrak{g}_{\mathbf{F}_3} := \mathfrak{g}(A_2)_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_3$ because m is not a multiple of 3. Recall from [Rou12, Proposition 3.2] that U^{ma+} can be expressed as a semidirect product of the form $U^{ma+} = (U_{\alpha_1} \times U_{\alpha_2}) \ltimes V$ for some normal subgroup V of U^{ma+} containing U^+_{im}. The fact that $\mathrm{ad}(f_1)x$ is nonzero in $\mathfrak{g}_{\mathbf{F}_3}$ then implies that by conjugating the imaginary root group $U_{(\delta)} \subset V$ by the root group $U_{-\alpha_1}$, one obtains a subset of U^{ma+} whose image in the quotient U_{α_1} of U^{ma+} is nontrivial (see [Rou12, 3.5] for the definition of the semidirect products defining the minimal parabolics in $\widehat{G}_2$). In particular, $U_{-\alpha_1}$ conjugates $U_{(\delta)} \subset U^+_{im}$ outside U^+_{im}, and hence U^+_{im} cannot be normal in $\widehat{G}_2$, a contradiction. □

6. Proof of Theorems A and B

We now let $k = \mathbf{F}_q$ be a finite field, A be an indecomposable generalised Cartan matrix of indefinite type, and we let G be one of the complete Kac–Moody groups $\mathfrak{G}^{rr}_A(\mathbf{F}_q)$ or $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$. We also set $U^+ := \mathfrak{U}^{rr+}(\mathbf{F}_q)$ or $U^+ := \mathfrak{U}^{ma+}(\mathbf{F}_q)$ accordingly. Then G is a locally compact totally disconnected topological group, and U^+ is a compact open subgroup of G. Indeed, for $\mathfrak{G}^{rr}_A(\mathbf{F}_q)$, this follows from [CR09, Proposition 1]; $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$ is locally compact because $\mathfrak{U}^{ma+}(\mathbf{F}_q)$ is compact open by [Rou12, 6.3], and it is totally disconnected because its filtration by the $\mathfrak{U}^{ma}_n(\mathbf{F}_q)$ is separated.

As mentioned in the introduction we first need to establish the topological simplicity of $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$ in full generality.

Proposition 6.1. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type. Then $\mathfrak{G}_A^{pma}(\mathbf{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$ is topologically simple over any finite field $\mathbf{F}_q$.*

Proof. Set $G := \mathfrak{G}_A^{pma}(\mathbf{F}_q)$ and $Z' := Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$. It follows from [CM11, Corollary 3.1] that G possesses a closed cocompact normal subgroup H containing Z' and such that H/Z' is topologically simple. It thus remains to see that in fact $H = G$. Let $\pi\colon G \to G/H$ denote the canonical projection. Let also w_1 be as in Theorem 4.8, and set $a := \overline{w_1} \in \mathfrak{N}(\mathbf{F}_q) \subset G$. Since G/H is compact and totally disconnected, its contraction groups are trivial (see e.g. the introduction to the Appendix). In particular,

$$\pi(\mathrm{con}(a^{\pm 1})) \subseteq \mathrm{con}(\pi(a^{\pm 1})) = \{1\},$$

and hence the closures of the contraction groups $\mathrm{con}(a)$ and $\mathrm{con}(a^{-1})$ are contained in $\ker \pi = H$. It follows from Theorem 4.8 that H contains $\mathfrak{U}^{ma+}(\mathbf{F}_q)$. But G normalises H and contains $\mathfrak{N}(\mathbf{F}_q)$, and hence H also contains all real root groups. Therefore $H = G$, as desired. ☐

We can now give a common proof for Theorems A and B:

Theorem 6.2. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type, and let G be one of the complete Kac–Moody groups $\mathfrak{G}_A^{rr}(\mathbf{F}_q)$ or $\mathfrak{G}_A^{pma}(\mathbf{F}_q)$. Then $G/Z'(G)$ is abstractly simple.*

Proof. Set $U^+ := \mathfrak{U}^{rr+}(\mathbf{F}_q)$ or $U^+ := \mathfrak{U}^{ma+}(\mathbf{F}_q)$ so that $U^+ \leq G$. Let K be a nontrivial normal subgroup of $G/Z'(G)$. Since $G/Z'(G)$ is topologically simple (see [CR09, Proposition 11] for $\mathfrak{G}_A^{rr}(\mathbf{F}_q)$ and Proposition 6.1 for $\mathfrak{G}_A^{pma}(\mathbf{F}_q)$), K must be dense in G. Since G is locally compact and totally disconnected, it then follows from Theorem 4.8 and Lemma 4.9, together with the Theorem from Appendix A, that K contains U^+. Since U^+ is open, K is open as well, and hence closed in G. Therefore $K = G$, as desired. ☐

Remark 6.3. We remark that, although we made the assumption that the Kac–Moody group $\mathfrak{G}(k)$ be of simply connected type (to get simplified statements), this of course does not have any impact on the results, and one might as well consider an arbitrary Kac–Moody root datum $\mathcal{D}$ and the Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(k)$. The essential difference is that, in general, $\mathfrak{G}_{\mathcal{D}}(k)$ is not generated by its root subgroups anymore, and one then has to consider a subquotient of $\mathfrak{G}^{pma}(k)$ (or else $\mathfrak{G}^{rr}(k)$). More precisely, let G be either $\mathfrak{G}^{pma}(k)$ or $\mathfrak{G}^{rr}(k)$, let U^+ be the corresponding subgroup $\mathfrak{U}^{ma+}(k)$ or $\mathfrak{U}^{rr+}(k)$, and let $G_{(1)}$ be the subgroup of G generated by U^+ and by all root groups of $\mathfrak{G}(k)$. Then $G_{(1)}$ is normal in G (and $G = \mathfrak{T}(k).G_{(1)}$), and what we proved is the abstract simplicity of $G_{(1)}/(Z'(G) \cap G_{(1)})$.

Appendix A. Contraction groups in normal closures

by Pierre-Emmanuel Caprace, Colin D. Reid and George A. Willis

Let G be a locally compact group. Given $f \in G$, we denote by $\mathrm{con}(f)$ the **contraction group** of the element f, which is defined as

$$\mathrm{con}(f) = \{g \in G \mid \lim_{n\to\infty} f^n g f^{-n} = 1\}.$$

It is indeed a subgroup of G, which need however not be closed in general. In case G is totally disconnected, Baumgartner and Willis [BW04] have characterised

the elements f with trivial contraction group as those whose conjugation action preserves a basis of identity neighbourhoods. In particular $\mathrm{con}(f) = 1$ if f is contained in some open compact subgroup of G, while $\mathrm{con}(f)$ is necessarily non-trivial if f does not normalise any open compact subgroup. The following result is thus empty in case G is a profinite group, but provides otherwise relevant information on abstract (potentially dense) normal subgroups.

Theorem. *Let G be a totally disconnected locally compact group and let $f \in G$. Any abstract normal subgroup of G containing f also contains the closure $\overline{\mathrm{con}(f)}$.*

The proof relies notably on some results of Baumgartner–Willis from [BW04]. We point out that, although the latter reference makes the hypothesis that the ambient group is metrisable, it was shown by Jaworski [Jaw09] that all the results remain valid without that assumption. We shall therefore freely refer to the results from [BW04] without any further comment on metrisability.

Proof of the Theorem. Let $H = \overline{\langle \mathrm{con}(f) \cup \{f\}\rangle}$. If $\overline{\langle f \rangle}$ is compact, then $\mathrm{con}(f)$ is trivial and there is nothing to prove. It may be supposed therefore that $\overline{\langle f \rangle}$ is not compact, in which case $\langle f \rangle$ is discrete and, furthermore, $\langle f \rangle \cap \mathrm{con}(f) = \{\mathrm{id}\}$.

Let U be a compact, open subgroup of H. Then, since f normalises $\mathrm{con}(f)$, the group $\langle \mathrm{con}(f) \cup \{f\}\rangle$ is isomorphic to $\langle f \rangle \ltimes \mathrm{con}(f)$. Moreover, any element of $\langle f \rangle \ltimes \mathrm{con}(f)$ with non-trivial image in the quotient $\langle f \rangle$ generates an infinite discrete cyclic subgroup of H, and can thus not belong to U. Therefore we have $U \cap \langle \mathrm{con}(f) \cup \{f\}\rangle \leq \mathrm{con}(f)$. Hence $U \leq \overline{\mathrm{con}(f)}$ and $\overline{\mathrm{con}(f)}$ is an open subgroup of H. We deduce that $H = \langle f \rangle \ltimes \overline{\mathrm{con}(f)}$. Let N be the (abstract) normal closure of f in H.

By [BW04, Corollary 3.30], we have $\overline{\mathrm{con}(f)} = \mathrm{nub}(f)\,\mathrm{con}(f)$, where $\mathrm{nub}(f)$ is defined as $\mathrm{nub}(f) = \overline{\mathrm{con}(f)} \cap \overline{\mathrm{con}(f^{-1})}$. By [BW04, Corollary 3.27], the group $\mathrm{nub}(f)$ is compact; by definition, it is normalised by f. Moreover, it follows from [Wil12, Proposition 4.8] that the conjugation $\langle f \rangle$-action on $\mathrm{nub}(f)$ is ergodic. We may thus invoke [Wil12, Proposition 7.1], which ensures that the map $\mathrm{nub}(f) \to \mathrm{nub}(f) : x \mapsto [f, x]$ is surjective. In particular the normal subgroup N contains $\mathrm{nub}(f)$.

We now invoke the Tree Representation Theorem from [BW04, Theorem 4.2]. This provides a locally finite tree T and a continuous homomorphism $\rho \colon H \to \mathrm{Aut}(T)$ enjoying the following properties:

- $\rho(f)$ acts as a hyperbolic isometry with attracting fixed point $\xi_+ \in \partial T$ and repelling fixed point $\xi_- \in \partial T$;
- $\rho(H)$ fixes ξ_- and is transitive on $\partial T \setminus \{\xi_-\}$;
- the stabiliser H_{ξ_+} coincides with $\mathrm{nub}(f) \rtimes \langle f \rangle$.

Any element $h \in H$ acting as a hyperbolic isometry fixes exactly two ends of T; one of them must thus be ξ_-. Since H is transitive on $\partial T \setminus \{\xi_-\}$, it follows that some conjugate of h is contained in H_{ξ_+}. We have seen that N contains $\mathrm{nub}(f) \rtimes \langle f \rangle = H_{\xi_+}$. We infer that N contains all elements of H acting as hyperbolic isometries on T.

Let now $\eta \in \partial T \setminus \{\xi_-\}$. There is some $h \in H$ such that $\rho(h).\xi_+ = \eta$. Using again the fact that $\rho(H)$ fixes ξ_-, we remark that if h is not a hyperbolic isometry, then hf is a hyperbolic isometry. Moreover we have $\rho(hf).\xi_+ = \eta$. Recalling that N contains all hyperbolic isometries of H, we infer that N is transitive on $\partial T \setminus \{\xi_-\}$. Therefore $N = H$ since N also contains H_{ξ_+}.

This proves that the normal closure of f in H contains $\overline{\mathrm{con}(f)}$. This implies a fortiori that the normal closure of f in G also contains $\overline{\mathrm{con}(f)}$. ☐

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